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General Construction of Nonstandard R_h -matrices as Contraction Limits of R_q -matrices

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Abstract

A class of transformations of R_q -matrices is introduced such that the $q \rightarrow 1$ limit gives explicit nonstandard R_h -matrices. The transformation matrix is singular itself at $q \rightarrow 1$ limit. For the transformed matrix, the singularities, however, cancel yielding a well-defined construction. Our method can be implemented systematically for R -matrices of all dimensions and not only for $sl(2)$ but also for algebras of higher dimensions. Explicit constructions are presented starting with $\mathcal{U}_q(sl(2))$ and $\mathcal{U}_q(sl(3))$, while choosing R_q for $(fund. \text{ rep.}) \otimes (arbitrary \text{ irrep.})$. The treatment for the general case and various perspectives are indicated. Our method yields nonstandard deformations along with a nonlinear map of the h -Borel subalgebra on the corresponding classical Borel subalgebra. For $\mathcal{U}_h(sl(2))$ this map is extended to the whole algebra and compared with another one proposed by us previously.

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The R -matrices for the fundamental representations of the nonstandard h -deformations of $sl(2)$ and $so(4)(\simeq sl(2) \otimes sl(2))$ were obtained [1,2] through a specific contraction of the corresponding q -deformed R -matrices. A similarity transformation of the 4×4 R_q -matrix for the fundamental representation of $\mathcal{U}_q(sl(2))$ was performed using a transforming matrix singular itself at the $q \rightarrow 1$ limit, but in such way that all singularities cancel out for the transformed R -matrix giving the finite nonstandard R_h -matrix. Following the previous practice [1,2], we refer to this combined process of similarity transformation and subsequent cancellation of singularities at the $q \rightarrow 1$ limit as contraction procedure. This technique was generalized to higher dimensional algebras [3] considering again the $N^2 \times N^2$ dimensional R -matrices for the fundamental representations of q -deformed $sl(N)$, for example. Other relevant references can be found in [1-3].

We present here an operatorial generalization of this approach directly applicable to R -matrices of arbitrary dimensions. For brevity and simplicity we start with $(\frac{1}{2} \otimes j)$ representation i.e. $2(2j+1) \times 2(2j+1)$ dimensional R_q -matrix for $\mathcal{U}_q(sl(2))$. Then we will indicate possible generalizations in different directions, using $\mathcal{U}_q(sl(3))$ as a particular example. The universal \mathcal{R} -matrix for $\mathcal{U}_h(sl(2))$ has been given a particularly convenient form [4,5]. For $(\frac{1}{2} \otimes j)$ representation this reduces to

$$R_h = \begin{pmatrix} e^{hX} & -hH + \frac{h}{2}(e^{hX} - e^{-hX}) \\ 0 & e^{-hX} \end{pmatrix}. \quad (1)$$

Here (H, X) are the generators of the Borel subalgebra of $\mathcal{U}_h(sl(2))$ satisfying

$$[H, X] = 2 \frac{\sinh hX}{h}. \quad (2)$$

Absent from the upper triangular form (1) and indeed from the universal \mathcal{R}_h -matrix is the generator Y completing the $\mathcal{U}_h(sl(2))$ algebra, namely satisfying

$$[H, Y] = -Y(\cosh hX) - (\cosh hX)Y, \quad [X, Y] = H. \quad (3)$$

We will show how (1) can be obtained, directly and for arbitrary j , from the corresponding R_q -matrix for $(\frac{1}{2} \otimes j)$ representation given by (see, for example [6])

$$R_q = \begin{pmatrix} q^{\mathcal{J}_0/2} & q^{1/2}(1 - q^{-2})\mathcal{J}_- \\ 0 & q^{-\mathcal{J}_0/2} \end{pmatrix}. \quad (4)$$

Here the generators of $\mathcal{U}_q(sl(2))$ are denoted by $(q^{\pm \mathcal{J}_0}, \mathcal{J}_{\pm})$ satisfying the standard relations

$$q^{\mathcal{J}_0} \mathcal{J}_{\pm} = \mathcal{J}_{\pm} q^{\mathcal{J}_0 \pm 2}, \quad [\mathcal{J}_+, \mathcal{J}_-] = \frac{q^{\mathcal{J}_0} - q^{-\mathcal{J}_0}}{q - q^{-1}} \equiv [\mathcal{J}_0]. \quad (5)$$

Throughout we consider generic q , excluding roots of unity.

For the purpose of transforming the R_q matrix in (4), we now consider a q -deformed exponential operator:

$$E_q(\eta\mathcal{J}_+) = \sum_{n=0}^{\infty} \frac{(\eta\mathcal{J}_+)^n}{[n]!}, \quad (6)$$

where

$$[n] \equiv \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]! = [n][n-1] \cdots [1], \quad [0]! \equiv 1.$$

We choose, for an arbitrary finite constant h , the parameter η as

$$\eta = \frac{h}{q-1}. \quad (7)$$

We emphasize that though the deformed exponential $E_q(x)$ defined in (6) has no convenient simple expression for its inverse (comparable to the standard q -exponential [7] satisfying $(\exp_q(x))^{-1} = \exp_{q^{-1}}(-x)$), this is precisely what is needed for obtaining non-singular limiting forms for the R -matrix elements for arbitrary representations and other interesting properties. For any given value of j , the series (6) may be terminated after setting $\mathcal{J}_+^{2j+1} = 0$; but, we proceed quite generally as follows. Defining

$$\mathcal{T}_{(\alpha)} = (E_q(\eta\mathcal{J}_+))^{-1} E_q(q^\alpha \eta\mathcal{J}_+) \quad (8)$$

with $\mathcal{T}_{(0)} = 1$, we obtain

$$(E_q(\eta\mathcal{J}_+))^{-1} q^{\frac{\alpha\mathcal{J}_0}{2}} E_q(\eta\mathcal{J}_+) = \mathcal{T}_{(\alpha)} q^{\frac{\alpha\mathcal{J}_0}{2}}. \quad (9)$$

For transforming the R_q matrix in (4), the operators $\mathcal{T}_{(\pm 1)}$ are of particular importance. We will be concerned with simple rational values of α . For later use, we note the identity

$$(E_q(\eta\mathcal{J}_+))^{-1} q^{\frac{(\alpha+\beta)\mathcal{J}_0}{2}} E_q(\eta\mathcal{J}_+) = \left((E_q(\eta\mathcal{J}_+))^{-1} q^{\frac{\alpha\mathcal{J}_0}{2}} E_q(\eta\mathcal{J}_+) \right) \left((E_q(\eta\mathcal{J}_+))^{-1} q^{\frac{\beta\mathcal{J}_0}{2}} E_q(\eta\mathcal{J}_+) \right),$$

which in notation (8) reads

$$\mathcal{T}_{(\alpha+\beta)} q^{\frac{\alpha+\beta}{2}\mathcal{J}_0} = (\mathcal{T}_{(\alpha)} q^{\frac{\alpha}{2}\mathcal{J}_0}) (\mathcal{T}_{(\beta)} q^{\frac{\beta}{2}\mathcal{J}_0}). \quad (10)$$

Moreover, using the identity

$$[\mathcal{J}_+^n, \mathcal{J}_-] = \frac{[n]}{q - q^{-1}} \left(q^{\mathcal{J}_0/2} \mathcal{J}_+^{n-1} q^{\mathcal{J}_0/2} - q^{-\mathcal{J}_0/2} \mathcal{J}_+^{n-1} q^{-\mathcal{J}_0/2} \right) \quad (11)$$

the following commutator is obtained

$$\begin{aligned} [E_q(\eta\mathcal{J}_+), \mathcal{J}_-] &= \frac{\eta}{q - q^{-1}} \left(q^{\mathcal{J}_0/2} E_q(\eta\mathcal{J}_+) q^{\mathcal{J}_0/2} - q^{-\mathcal{J}_0/2} E_q(\eta\mathcal{J}_+) q^{-\mathcal{J}_0/2} \right) \\ &= \frac{\eta}{q - q^{-1}} \left(E_q(q\eta\mathcal{J}_+) q^{\mathcal{J}_0} - E_q(q^{-1}\eta\mathcal{J}_+) q^{-\mathcal{J}_0} \right), \end{aligned} \quad (12)$$

which, in turn, leads to

$$(E_q(\eta\mathcal{J}_+))^{-1} \mathcal{J}_- E_q(\eta\mathcal{J}_+) = -\frac{\eta}{q - q^{-1}} \left(\mathcal{T}_{(1)} q^{\mathcal{J}_0} - \mathcal{T}_{(-1)} q^{-\mathcal{J}_0} \right) + \mathcal{J}_-. \quad (13)$$

Evaluating term by term, the $q \rightarrow 1$ limits of $\mathcal{T}_{(\pm 1)}$ are found to be *finite* and of the form

$$\lim_{q \rightarrow 1} \mathcal{T}_{(\pm 1)} = T_{(\pm 1)} = \sum_{n=0}^{\infty} c_n^{(\pm)} (hJ_+)^n, \quad (14)$$

where (J_0, J_{\pm}) are the generators of the classical $sl(2)$ algebra:

$$[J_0, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = J_0. \quad (15)$$

The first few coefficients $\{c_n^{(\pm)} \mid n = 0, 1, 2, \dots\}$ in (14) read

$$c_0^{(\pm)} = 1, \quad c_1^{(\pm)} = \pm 1, \quad c_2^{(\pm)} = \frac{1}{2}, \quad c_3^{(\pm)} = 0, \quad c_4^{(\pm)} = -\frac{1}{8}, \quad c_5^{(\pm)} = 0. \quad (16)$$

The series $(E_q(x))^{-1}$ can be constructed systematically upto any given order in η ; and, consequently, the operators $T_{(\pm 1)}$ are obtained. Along these lines a program in MAPLE for evaluation of the coefficients $c_n^{(\pm)}$ is easy to make. If the limits are indeed finite, then from (10) it is evident that

$$T_{(\alpha)} = (T_{(1)})^{\alpha}, \quad (17)$$

where $T_{(\alpha)} = \lim_{q \rightarrow 1} \mathcal{T}_{(\alpha)}$. Henceforth we write

$$T_{(1)} = T. \quad (18)$$

The result (16) suggests the following derivation of the *closed* form of T . To this end, we left and right multiply the second commutation relation in (5) by $(E_q(\eta\mathcal{J}_0))^{-1}$ and $E_q(\eta\mathcal{J}_0)$ respectively:

$$(E_q(\eta\mathcal{J}_+))^{-1} (q^{\mathcal{J}_0} - q^{-\mathcal{J}_0}) E_q(\eta\mathcal{J}_+) = (q - q^{-1}) (E_q(\eta\mathcal{J}_+))^{-1} [\mathcal{J}_+, \mathcal{J}_-] E_q(\eta\mathcal{J}_+). \quad (19)$$

Using (7), (9) and (13), we obtain

$$\begin{aligned} \mathcal{T}_{(2)} q^{\mathcal{J}_0} - \mathcal{T}_{(-2)} q^{-\mathcal{J}_0} &= -\frac{h}{q - 1} \left(\mathcal{T}_{(1)} (\mathcal{J}_+ q^{\mathcal{J}_0} - q^{\mathcal{J}_0} \mathcal{J}_+) - \mathcal{T}_{(-1)} (\mathcal{J}_+ q^{-\mathcal{J}_0} - q^{-\mathcal{J}_0} \mathcal{J}_+) \right) \\ &\quad + (q - q^{-1}) [\mathcal{J}_+, \mathcal{J}_-] \\ &= h(q + 1) \left(\mathcal{T}_{(1)} \mathcal{J}_+ q^{\mathcal{J}_0} + q^{-2} \mathcal{T}_{(-1)} \mathcal{J}_+ q^{-\mathcal{J}_0} \right) + (q^{\mathcal{J}_0} - q^{-\mathcal{J}_0}). \end{aligned} \quad (20)$$

Using (17), we now obtain the following equation for T at the limit $q \rightarrow 1$:

$$T^2 - T^{-2} = (T + T^{-1})(2hJ_+),$$

which after a factorization yields

$$T - T^{-1} = 2hJ_+. \quad (21)$$

The quadratic relation (21) in T is now solved:

$$T^{\pm 1} = \pm hJ_+ + \left(1 + (hJ_+)^2\right)^{1/2}. \quad (22)$$

This is our crucial result.

With all these results now in hand we go back to R_q in (4). We choose the transformation matrix as $G = g_{1/2} \otimes g$, where $g = E_q(\eta\mathcal{J}_+)$ and $g_{1/2} \equiv g|_{j=\frac{1}{2}}$. The similarity transformation now yields

$$G^{-1}R_qG = \begin{pmatrix} g^{-1}q^{\mathcal{J}_0/2}g & \eta g^{-1}(q^{\mathcal{J}_0/2} - q^{-\mathcal{J}_0/2})g + q^{-1/2}(q - q^{-1})g^{-1}\mathcal{J}_-g \\ 0 & g^{-1}q^{-\mathcal{J}_0/2}g \end{pmatrix}. \quad (23)$$

Implementing the relevant results from (9) to (23), we now obtain

$$R_h = \lim_{q \rightarrow 1} (G^{-1}R_qG) = \begin{pmatrix} T & -\frac{h}{2}(T + T^{-1})J_0 + \frac{h}{2}(T - T^{-1}) \\ 0 & T^{-1} \end{pmatrix} \quad (24)$$

$$= \begin{pmatrix} e^{h\hat{X}} & -h\hat{H} + \frac{h}{2}(e^{h\hat{X}} - e^{-h\hat{X}}) \\ 0 & e^{-h\hat{X}} \end{pmatrix}, \quad (25)$$

where we have defined

$$e^{\pm h\hat{X}} = T^{\pm 1} = \pm hJ_+ + (1 + (hJ_+)^2)^{1/2} \quad (26)$$

$$\hat{H} = \frac{1}{2}(T + T^{-1})J_0 = (1 + (hJ_+)^2)^{1/2}J_0. \quad (27)$$

It is easily to verify that

$$[\hat{H}, T^{\pm 1}] = T^{\pm 2} - 1 \quad \Rightarrow \quad [\hat{H}, \hat{X}] = \frac{2}{h} \sinh(h\hat{X}). \quad (28)$$

Comparing (1), (2) with (25), (28) respectively we see that the contraction scheme, which comprised of our transformation and limiting procedure has furnished the R_h -matrix along

with a *nonlinear map of the Borel subalgebra of $\mathcal{U}_h(sl(2))$ generated by (H, X) on the classical one generated by (J_0, J_+)* . Indeed, defining

$$\hat{Y} = J_- - \frac{h^2}{4} J_+ (J_0^2 - 1) \quad (29)$$

we complete the $\mathcal{U}_h(sl(2))$ algebra and show that

$$[\hat{H}, \hat{Y}] = -(\hat{Y} \cosh(h\hat{X}) + \cosh(h\hat{X}) \hat{Y}), \quad [\hat{X}, \hat{Y}] = \hat{H}. \quad (30)$$

The triplet $(\hat{H}, \hat{X}, \hat{Y})$ may be looked as a particular realization of the $\mathcal{U}_h(sl(2))$ generators (H, X, Y) . We have, therefore developed an invertible nonlinear map of (H, X, Y) on classical generators (J_0, J_+, J_-) . The full coalgebra structure of $\mathcal{U}_h(sl(2))$ can also be implemented.

We have presented before [8,9] an alternative realization of the $\mathcal{U}_h(sl(2))$ generators in terms of the classical generators:

$$e^{h\tilde{X}} = \frac{I + \frac{h}{2}J_+}{I - \frac{h}{2}J_+} \quad \Rightarrow \quad \frac{h\tilde{X}}{2} = \operatorname{arctanh}\left(\frac{hJ_+}{2}\right), \quad (31)$$

whereas the remaining maps read

$$\tilde{Y} = (1 - (hJ_+/2)^2)^{1/2} J_- (1 - (hJ_+/2)^2)^{1/2}, \quad \tilde{H} = J_0. \quad (32)$$

Both the sets $(\hat{H}, \hat{X}, \hat{Y})$ and $(\tilde{H}, \tilde{X}, \tilde{Y})$ satisfy the same $\mathcal{U}_h(sl(2))$ algebra. Strictly speaking such realizations hold on finite dimensional vectors spaces of irreducible representations. The nilpotency of the classical J_+ then, via the construction of the triplets $(\hat{H}, \hat{X}, \hat{Y})$ and $(\tilde{H}, \tilde{X}, \tilde{Y})$, provides finite dimensional irreducible representations of the $\mathcal{U}_h(sl(2))$ algebra. Inverting the maps (26) and (31) we obtain

$$J_+ = \frac{1}{h} \sinh(h\hat{X}) = \frac{2}{h} \tanh\left(\frac{h\tilde{X}}{2}\right). \quad (33)$$

In (29), we obtain that \hat{Y} is nonlinear in J_0 . This is a consequence of the nonlinearity involved in the definition (27) of \hat{H} .

So far, as a relatively simple, illustrative example of our method, we have been considering the case $(\frac{1}{2} \otimes j)$. But our method can be used to obtain the universal \mathcal{R}_h -matrix [4,5] in the form

$$\mathcal{R}_h = \exp\left(-h\hat{X} \otimes e^{h\hat{X}} \hat{H}\right) \exp\left(h e^{h\hat{X}} \hat{H} \otimes \hat{X}\right). \quad (34)$$

In obtaining this form as the $q \rightarrow 1$ limit of transformed universal \mathcal{R} -matrix of $\mathcal{U}_q(sl(2))$ the cancellation of divergences become more subtle and complicated. We have fully analyzed it for the case $(1 \otimes j)$. The procedure will be presented elsewhere. But (34) must evidently have all the required properties.

We can study representations of $\mathcal{U}_q(sl(2))$ using the map $(\hat{H}, \hat{X}, \hat{Y}) \longrightarrow (J_0, J_\pm)$ as it was done in [8,9] using $(\tilde{H}, \tilde{X}, \tilde{Y}) \longrightarrow (J_0, J_\pm)$. This aspect will not be treated here. For the particularly simple example of $(\frac{1}{2} \otimes 1)$ representation, we compare

$$R_h(\hat{H}, \hat{X}, \hat{Y}) = \begin{pmatrix} \hat{A} & \hat{B} \\ 0 & \hat{C} \end{pmatrix} \quad (35)$$

with

$$R_h(\tilde{H}, \tilde{X}, \tilde{Y}) = \begin{pmatrix} \tilde{A} & \tilde{B} \\ 0 & \tilde{C} \end{pmatrix}. \quad (36)$$

For the representation $(\frac{1}{2} \otimes 1)$, $R_h(\tilde{H}, \tilde{X}, \tilde{Y})$ was already given in [8]:

$$\tilde{A} = \begin{pmatrix} 1 & 2h & 2h^2 \\ 0 & 1 & 2h \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} -2h & 2h^2 & 0 \\ 0 & 0 & 2h^2 \\ 0 & 0 & 2h \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} 1 & -2h & 2h^2 \\ 0 & 1 & -2h \\ 0 & 0 & 1 \end{pmatrix}. \quad (37)$$

Using our new mapping, we obtain

$$\hat{A} = \tilde{A}, \quad \hat{B} = \begin{pmatrix} -2h & 2h^2 & 4h^3 \\ 0 & 0 & 2h^2 \\ 0 & 0 & 2h \end{pmatrix}, \quad \hat{C} = \tilde{C}. \quad (38)$$

The two R -matrices (35) and (36) are related through a similarity transformation by a matrix of the form $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes M$, where

$$M = \begin{pmatrix} 1 & 0 & h^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (39)$$

The major interest in our method is that *it can be generalized for obtaining the nonstandard R -matrices of algebras of higher dimensions as contraction limits of the corresponding R_q -matrices.* Here we illustrate the scope of our method, using $sl(3)$ as an example. We treat the case of R -matrices corresponding to the representation $(fund.) \otimes (arb. irrep.)$. Possibilities of further generalizations are then indicated. Here we assume the $\mathcal{U}_q(sl(3))$ Hopf algebra to be well-known [6]. Let the standard Chevally generators of $\mathcal{U}_q(sl(3))$ be $(q^{\pm h_1}, q^{\pm h_2}, \hat{e}_1, \hat{e}_2, \hat{f}_1, \hat{f}_2)$, whereas the classical ($q = 1$) generators are denoted by $(h_1, h_2, e_1, e_2, f_1, f_2)$. Defining

$$\hat{e}_3 = \hat{e}_1 \hat{e}_2 - q^{-1} \hat{e}_2 \hat{e}_1, \quad \hat{f}_3 = \hat{f}_2 \hat{f}_1 - q \hat{f}_1 \hat{f}_2, \quad (40)$$

we obtain

$$\begin{aligned}
q^{h_i} \hat{e}_3 &= \hat{e}_3 q^{h_i+1} \quad \text{for } (i = 1, 2), & [\hat{e}_3, \hat{f}_3] &= [h_1 + h_2], \\
[\hat{e}_3, \hat{f}_1] &= -\hat{e}_2 q^{-h_1}, & [\hat{e}_3, \hat{f}_2] &= q^{h_2} \hat{e}_1, \\
\hat{e}_1 \hat{e}_3 &= q \hat{e}_3 \hat{e}_1, & \hat{e}_2 \hat{e}_3 &= q^{-1} \hat{e}_3 \hat{e}_2.
\end{aligned} \tag{41}$$

As was noted in [3], after using the contraction process for $\mathcal{U}_q(sl(2))$ subalgebra and possible relabeling of the generators, the really new possibility for $\mathcal{U}_q(sl(3))$ arises from using \hat{e}_3 in the construction of the operator performing the similarity transformation. More generally, similar new possibility for the $\mathcal{U}_q(sl(N))$ emerges by using the generators corresponding to the highest positive root for the same purpose. As a direct generalization of (6) we now consider

$$E_q(\eta \hat{e}_3) = \sum_{n=0}^{\infty} \frac{(\eta \hat{e}_3)^n}{[n]!}. \tag{42}$$

Using (41), we obtain

$$\begin{aligned}
\hat{e}_1 E_q(\eta \hat{e}_3) &= E_q(q \eta \hat{e}_3) \hat{e}_1, & \hat{e}_2 E_q(\eta \hat{e}_3) &= E_q(q^{-1} \eta \hat{e}_3) \hat{e}_2, \\
[E_q(\eta \hat{e}_3), f_1] &= -\eta E_q(q^{-1} \eta \hat{e}_3) \hat{e}_2 q^{-h_1} & [E_q(\eta \hat{e}_3), f_2] &= \eta E_q(q \eta \hat{e}_3) q^{h_2} \hat{e}_1 \\
[E_q(\eta \hat{e}_3), f_3] &= \frac{\eta}{q - q^{-1}} \left(E_q(q \eta \hat{e}_3) q^{h_1+h_2} - E_q(q^{-1} \eta \hat{e}_3) q^{-h_1-h_2} \right)
\end{aligned} \tag{43}$$

Analogous to the example of $\mathcal{U}_q(sl(2))$ algebra, we here define

$$\hat{T}_{(\alpha)} = (E_q(\eta \hat{e}_3))^{-1} E_q(q^\alpha \eta \hat{e}_3), \tag{44}$$

$$T_{(\alpha)} = \lim_{q \rightarrow 1} \hat{T}_{(\alpha)}. \tag{45}$$

To obtain the finite limit of the operators $T_{(\alpha)}$ we may proceed by exploiting the $\mathcal{U}_q(sl(2))$ subalgebra generated by $(e_3, f_3, q^{\pm(h_1+h_2)})$. Equivalently, we note that the *in the formal series development of $E_q(x)$, as well as in the subsequent limiting process no specific properties of \mathcal{J}_+ (or \hat{e}_3) need to be introduced.* Parallel to (22) we here obtain

$$T_{(\pm 1)} = \pm h e_3 + \left(1 + (h e_3)^2 \right)^{1/2}, \quad T_{(\alpha)} = T^\alpha. \tag{46}$$

For $\mathcal{U}_q(sl(3))$ algebra the R_q matrix in the representation $(fund.) \otimes (arbitrary \text{ irrep.})$ reads [6]:

$$R_q = \begin{pmatrix} q^{\frac{1}{3}(2h_1+h_2)} & q^{\frac{1}{3}(2h_1+h_2)} \Lambda_{12} & q^{\frac{1}{3}(2h_1+h_2)} \Lambda_{13} \\ 0 & q^{-\frac{1}{3}(h_1-h_2)} & q^{-\frac{1}{3}(h_1-h_2)} \Lambda_{23} \\ 0 & 0 & q^{-\frac{1}{3}(h_1+2h_2)} \end{pmatrix}, \tag{47}$$

where

$$\begin{aligned}\Lambda_{12} &= q^{-1/2}(q - q^{-1})q^{-h_1/2}\hat{f}_1, & \Lambda_{23} &= q^{-1/2}(q - q^{-1})q^{-h_2/2}\hat{f}_2, \\ \Lambda_{13} &= q^{-1/2}(q - q^{-1})\hat{f}_3q^{-\frac{1}{2}(h_1+h_2)}.\end{aligned}\quad (48)$$

We introduce the operator

$$\mathbf{G} = E_q(\eta\hat{e}_3)_{(fund.)} \otimes E_q(\eta\hat{e}_3)_{(arb.)} \quad (49)$$

and perform similarity transformation on the R_q matrix (47)

$$\mathbf{G}^{-1}R_q\mathbf{G} = \begin{pmatrix} \mathbf{g}^{-1}q^{\frac{1}{3}(2h_1+h_2)}\mathbf{g} & \mathbf{g}^{-1}q^{\frac{1}{3}(2h_1+h_2)}\Lambda_{12}\mathbf{g} & \eta\mathbf{g}^{-1}(q^{\frac{1}{3}(2h_1+h_2)} - q^{-\frac{1}{3}(h_1+2h_2)})\mathbf{g} \\ & & + \mathbf{g}^{-1}q^{\frac{1}{3}(2h_1+h_2)}\Lambda_{13}\mathbf{g} \\ 0 & \mathbf{g}^{-1}q^{-\frac{1}{3}(h_1-h_2)}\mathbf{g} & \mathbf{g}^{-1}q^{-\frac{1}{3}(h_1-h_2)}\Lambda_{23}\mathbf{g} \\ 0 & 0 & \mathbf{g}^{-1}q^{-\frac{1}{3}(h_1+2h_2)}\mathbf{g} \end{pmatrix} \quad (50)$$

where we use the notation $\mathbf{g} = E_q(\eta\hat{e}_3)$. In close analogy with earlier example, the transform turns out to be finite at the $q \rightarrow 1$ limit:

$$\lim_{q \rightarrow 1} \mathbf{G}^{-1}R_q\mathbf{G} = R_h = \begin{pmatrix} T & 2hT^{-1/2}e_2 & -\frac{h}{2}(T + T^{-1})(h_1 + h_2) + \frac{h}{2}(T - T^{-1}) \\ 0 & I & -2hT^{1/2}e_1 \\ 0 & 0 & T^{-1} \end{pmatrix}. \quad (51)$$

We have, briefly, directly stated the final result. But the derivation is fairly analogous to $sl(2)$ case. *In fact this illustrates how directly our formalism can be implemented for higher dimensional cases.*

We note that the corner elements of (51) have exactly the same structure as (24). The new features arises with the presence of terms $2hT^{-1/2}e_2$, $-2hT^{1/2}e_1$ involving the simple root generators. In general, we may proceed as follows. The matrix structure in (51) provides a realization of the Lax operator corresponding to the h -Borel subalgebra $\mathcal{U}_h(\mathcal{B}_+)$ generated by the Cartan elements and the positive root generators. In fact following the analogy for the $\mathcal{U}_h(sl(2))$ case, as evidenced by comparing (1) with (25), the matrix operator (51) provides a map of the h -Borel subalgebra $\mathcal{U}_h(\mathcal{B}_+)$ on its classical counterpart. The full Hopf structure of the h -Borel subalgebra may then be obtained by the standard FRT procedure [10]. In the instance of nonstandard h -deformed $sl(2)$ algebra, the universal R_h -matrix for this Borel subalgebra is the universal R_h -matrix for the full $\mathcal{U}_h(sl(2))$ Hopf algebra. Assuming that this property still holds for all h -deformed $sl(N)$ algebra, then our construction may provide a route to obtain the so far uninvestigated $\mathcal{U}_h(sl(N))$ algebras for $N > 2$. In this instance, it is worth pointing out the close kinship of the h -deformed algebras with κ -Poincaré and

deformed conformal algebras [11], where the deformation parameter has dimensions of mass and, consequently, an induced fundamental length scale enters at the geometrical level.

To elucidate a further possible application of our present technique of extraction of h -deformed objects from the corresponding q -deformed objects via contraction procedure, we return to the earlier example of deformed $sl(2)$ algebra. Exploiting the bialgebraic duality relationship between the pair of Hopf algebras $\mathcal{F}un_q(GL(2))$ and $\mathcal{U}_q(gl(2))$, the dual form, which provides a generalization of the familiar exponential relationship between classical Lie groups and algebras, was constructed in [12]. This may be used to provide [13] arbitrary finite dimensional irreducible representations of $GL_q(2)$ leading to the q -analogues of the classical Wigner $d^{(j)}$ -functions or spherical functions. The present method of using similarity transformation, singular itself at $q \rightarrow 1$ limit, but in such a way that the singularities get cancelled to provide finite results for the transformed objects, may then be used to obtain arbitrary finite dimensional representations of the h -deformed group elements $GL_h(2)$ and, consequently, h -deformed analogues of Wigner $d^{(j)}$ functions. We will return to this topic later.

At the end we wish to mention a fact as a curiosity. There exist canonical transformations of the classical Heisenberg algebra closely mimicking our construction of the nonlinear maps of $\mathcal{U}_h(sl(2))$ generators on the generators of the classical $sl(2)$ algebra. Starting with classical Heisenberg commutation relation $[a_-, a_+] = 1$, we define an one-parameter (h) family of transformations

$$\hat{a}_+ = \frac{1}{h} \ln(ha_+ + \sqrt{1 + h^2 a_+^2}), \quad \hat{a}_- = \sqrt{1 + h^2 a_+^2} \ a_-, \quad (52)$$

which satisfy Heisenberg commutation relation $[\hat{a}_-, \hat{a}_+] = 1$. This closely resembles the nonlinear realization of the generators of $\mathcal{U}_h(sl(2))$ discussed in (26), (27) and (28). There exists also an analogous family of canonical transformations closely paralleling the other nonlinear realization of $\mathcal{U}_h(sl(2))$ generators discussed in (31) and (32):

$$\tilde{a}_+ = \frac{2}{h} \operatorname{arctanh}\left(\frac{ha_+}{2}\right), \quad \tilde{a}_- = \sqrt{1 - \frac{h^2 a_+^2}{4}} \ a_- \ \sqrt{1 - \frac{h^2 a_+^2}{4}}, \quad (53)$$

which again satisfies the commutation relation $[\tilde{a}_-, \tilde{a}_+] = 1$. It is perhaps worth investigating the possible connection between the theory of canonical transformations of the Heisenberg algebra and the nonlinear map discussed above.

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